Recitation 10/8 Solutions

1. Let $V = \left\{ \begin{bmatrix} 2x - y + z \\ 3x + z \\ y - z \\ z \end{bmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}$.

   (a) Show that V is a subspace of $\mathbb{R}^4$ by showing that is the span of three vectors.

   Vectors in V are of the form $\begin{bmatrix} 2x - y + z \\ 3x + z \\ y - z \\ z \end{bmatrix} = x \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

   Hence $V = \text{Span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

   (b) Interpret V as the column space of a matrix and use this matrix to determine $V^{\perp}$.

   We can view V as the column space of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Since the orthogonal complement of $C(A)$ is $N(A^T)$, we need to find a basis for $N(A^T)$. Using the function "nullspace(A')" in Julia provides a basis (although different than the one we typically find by hand). We can also find the basis by hand as follows.

   The RREF of A^T (using Julia command "rref(A')") is $\begin{bmatrix} 1 & 0 & 0 & -1.5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1.5 \end{bmatrix}$. The corresponding system is

$$\begin{aligned} x_1 - 1.5x_4 &= 0 \\ x_2 + x_4 &= 0 \\ x_3 - 1.5x_4 &= 0 \end{aligned}$$

   Solving in terms of the free variable x_4 gives the general solution

$$\mathbf{x} = \begin{bmatrix} 1.5x_4 \\ -x_4 \\ 1.5x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1.5 \\ -1 \\ 1.5 \\ 1 \end{bmatrix}.$$

Hence $V^\perp = N(A^T) = \text{Span}\left\{ \begin{bmatrix} 1.5 \\ -1 \\ 1.5 \\ 1 \end{bmatrix} \right\}$.

2. Let V be a subspace of $\mathbb{R}^m$. Show that $\dim V + \dim V^\perp = m$ by viewing V as the column space of a matrix and using facts you know about the four fundamental subspaces.

   Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a basis for V. Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$; note that A is $m \times n$. Then $V = C(A)$. Now the orthogonal complement of $C(A)$ is $N(A^T)$. We know that $\dim C(A) = \text{rank}A$ and $\dim N(A^T) = m - \text{rank}A$. Hence $\dim V + \dim V^\perp = \dim C(A) + \dim N(A^T) = \text{rank}A + m - \text{rank}A = m$.

3. Let $W = \text{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 2 \\ 1 \end{bmatrix} \right\}$ and let $\mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \\ 2 \end{bmatrix}$

   (a) Show that the three vectors in the description of W form a basis for W.

   The RREF of the matrix $A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -4 \\ 3 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Since there is a pivot in each column, the columns of A are linearly independent. Since these columns span W, they form a basis.

   (b) Find the projection of $\mathbf{b}$ onto W. Interpret your answer geometrically

   First we compute $A^TA = \begin{bmatrix} -1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -4 \\ 3 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -1 & -3 \\ -1 & 3 & -5 \\ -3 & -5 & 22 \end{bmatrix}$

   and $A^T\mathbf{b} = \begin{bmatrix} -1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ 14 \end{bmatrix}$ (via Julia).

   Next, we solve the system $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$ (via Julia) to get the solution: $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Now the projection is $\text{proj}_W \mathbf{b} = A\hat{\mathbf{x}} = \begin{bmatrix} 0 \\ -1 \\ 4 \\ 2 \end{bmatrix}$. But this is just $\mathbf{b}$! Since projecting $\mathbf{b}$ did not change $\mathbf{b}$, we have that $\mathbf{b} \in W$.

(c) What is the projection matrix P associated to this projection? Compute it using Julia and use it to verify (in Julia) your answer to part (b) is correct.

$P = A(A^TA)^{-1}A^T$, which can be computed using Julia. Using Julia, we see that $P\mathbf{b} = \mathbf{b}$.